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Primitive 2-factorizations of the complete graph

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Abstract

Let $\mathscr{F}$ be a 2-factorization of the complete graph K_v admitting an automorphism group G acting primitively on the set of vertices. If $\mathscr{F}$ consists of Hamiltonian cycles, then $\mathscr{F}$ is the unique, up to isomorphisms, 2-factorization of K_{p^n} admitting an automorphism group which acts 2-transitively on the vertex-set, see [A. Bonisoli, M. Buratti, G. Mazzuoccolo, Doubly transitive 2-factorizations, J. Combin. Designs 15 (2007) 120–132.]. In the non-Hamiltonian case we construct an infinite family of examples whose automorphism group does not contain a subgroup acting 2-transitively on vertices.

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1. Introduction

For an integer $v \geqslant 3$, let K_v be the complete (simple undirected) graph on v vertices with vertex-set $V(K_v)$ and edge set $E(K_v)$. For $3 \leqslant k \leqslant v$, a *k-cycle* $C = (x_0, x_1, \ldots, x_{k-1})$ is the subgraph of K_v whose edges are $[x_i, x_{i+1}]$, $i = 0, \ldots, k-1$, indices taken modulo k. If $k = v$, the cycle is called Hamiltonian.

A 2-*factor* F of K_v is a set of cycles whose vertices partition $V(K_v)$. A 2-*factorization* of K_v is a set $\mathscr{F}$ of edge disjoint 2-factors forming a cover of $E(K_v)$. A 2-factorization in which all the 2-factors are isomorphic to a factor F is called an *F-factorization*. If each 2-factor of $\mathscr{F}$ consists of a single Hamiltonian cycle, $\mathscr{F}$ is called a *Hamiltonian 2-factorization*. The existence of a 2-factorization of K_v forces v to be odd.

The collection of cycles appearing in the factors of $\mathscr{F}$ form a cycle decomposition of K_v, which is called the underlying decomposition. We will denote it by $\mathscr{D}_{\mathscr{F}}$.

A permutation group G acting faithfully on $V(K_v)$ and preserving the 2-factorization $\mathscr{F}$ is called an *automorphism group* of $\mathscr{F}$.

In some recent papers the possible structures and actions of G on vertices or factors have been investigated. In [3] the situation in which G acts regularly (i.e. sharply transitively) on vertices is studied in detail. In [2] a complete description of G and $\mathscr{F}$ is given in case the action of G is doubly transitive on the vertex-set. In particular it is proved that if $\mathscr{F}$ is Hamiltonian, then v is an odd prime p, the group G is the affine general linear group $\mathrm{AGL}(1, p)$ and, if the vertices of

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K_p are labelled by the elements of $\mathbb{Z}_p$, then $\mathscr{F} = \{C_1, C_2, \ldots, C_{(p-1)/2}\}$, with $C_i = (0, i, 2i, \ldots, (p-1)i)$ (subscripts mod p), $i = 1, 2, \ldots, (p-1)/2$. This factorization is the *natural* 2-factorization (also denoted by $\mathscr{N}(\mathbb{Z}_p)$) which arises from the cyclic group $\mathbb{Z}_p$, see [3].

In this paper, we investigate primitive 2-factorizations, i.e. admitting an automorphism group G with primitive action on the vertex-set. Note that all 2-factorizations admitting an automorphism group doubly transitive on the vertex-set are also examples of primitive 2-factorizations.

If $\mathscr{F}$ is Hamiltonian, we prove that v is an odd prime p and $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$. Moreover, the group G is necessarily a subgroup of $\mathrm{AGL}(1, p)$ containing $\mathbb{Z}_p$. In the non-Hamiltonian case, we give examples of primitive 2-factorizations whose full automorphism group does not contain a subgroup acting doubly transitively on the vertices. In the last section we also prove that a primitive 2-factorization of K_9 is necessarily 2-transitive, whence a 2-factorization arising from the affine line parallelism of $\mathrm{AG}(2, 3)$ in a suitable manner, see [2], and a primitive 2-factorization of K_{15} does not exist.

2. The Hamiltonian case

In this section we prove that $\mathscr{N}(\mathbb{Z}_p)$ is the unique primitive Hamiltonian 2-factorization of a complete graph.

Lemma 1. *Let $\mathscr{F}$ be a 2-factorization of K_p with a transitive automorphism group G. Then $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$ and $G \leqslant \mathrm{AGL}(1, p)$.*

Proof. By transitivity of G on $V(K_p)$, the integer p is a divisor of the order of G, then an element $g \in G$ of order p exists. The cyclic group generated by g acts regularly on the vertex-set. By proposition 2.8 of [3], it is $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$. The full group of automorphism of $\mathscr{F}$ is $\mathrm{AGL}(1, p)$ (see [2], Section 1) and then $G \leqslant \mathrm{AGL}(1, p)$. $\square$

Theorem 1. *Let $\mathscr{F}$ be a Hamiltonian 2-factorization of K_v with primitive automorphism group G. Then $v = p$, $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$ and $\mathbb{Z}_p \leqslant G \leqslant \mathrm{AGL}(1, p)$.*

Proof. Suppose G is of even order. We prove that G contains exactly v involutions. First of all observe that each involution of G fixes all the 2-factors of $\mathscr{F}$. In fact let $g \in G$ be an involution exchanging two vertices x_0 and x_1. Labelling the vertices such that $C = (x_0, x_1, \ldots, x_{v-1})$ is the unique cycle of $\mathscr{D}_{\mathscr{F}}$ containing $[x_0, x_1]$, we obtain:

$$g(x_i) = x_{v+1-i}, \quad g(x_{v+1-i}) = x_i \quad \text{for } i = 1, \ldots, \frac{v-1}{2},$$
$$g(x_{(v+1)/2}) = x_{(v+1)/2},$$

where all indices are taken mod v. Then g fixes the vertex $x_{(v+1)/2}$ and each edge of the set $E = \{[x_i, x_{v+1-i}]/i = 1, \ldots, (v-1)/2\}$. Suppose that at least two edges of E belong to the same 2-factor F of $\mathscr{F}$. Then g should fix the unique cycle of F and two edges on it: a contradiction. We have proved that the elements in E are in different 2-factors. By the fact that the cardinality of E coincides with the number of 2-factors, we conclude that g fixes all the factors of $\mathscr{F}$. Furthermore we can also observe that each involution in G fixes exactly one vertex of $V(K_v)$. Let now $x \in V(K_v)$, we have $|G| = |G_x|v$, where G_x is the stabilizer of the vertex x, then $|G_x|$ is even and we have at least one involution in G_x. Observe that G_x contains exactly one involution; in fact if we fix a 2-factor F and C is its cycle, the action of an involution of G_x is uniquely determined by its action on the vertices of C as above explained. We can conclude that the group G contains exactly v distinct involutions. In particular for each factor F, the subgroup G_F contains v involutions: namely all the involutions of G. It is well known that $G_F \leqslant D_v$, the dihedral group on v vertices, and then $G_F \cong D_v$ for each $F \in \mathscr{F}$. Furthermore, for each factor $F' \in \mathscr{F} - \{F\}$, the dihedral groups G_F and G'_F contain exactly the same v involutions, therefore $G_F = G'_F$. Label the vertices of K_v by the elements $0, 1, \ldots, v-1$ in such a way that a 2-factor $F \in \mathscr{F}$ contains the cycle $(0, 1, \ldots, v-1)$and an element of G_F of order v maps the vertex i onto $i+1$, for each i. Denote by $g \in G_F$ this element. Suppose v is not a prime and let h be a proper divisor of v. Let $F' \in \mathscr{F}$ be the 2-factor containing the edge $[0, h]$. As $G_{F'} = G_F$ we have $g^h \in G_{F'}$ and then F' contains the cycle $(0, h, 2h, \ldots, v-h)$ of length less than v: a contradiction. We conclude that v is a prime. By Lemma 1 the assertion follows in this case. We have proved that for a primitive Hamiltonian 2-factorization of K_v only two possibilities hold: either v is a prime or $|G|$ is odd. By the O'Nan Scott Theorem (see [6]) and by the fact that a simple non-abelian group has even order (Feit–Thomson Theorem, see [7]), the socle of G is a regular elementary abelian p-group for some prime p, $v = p^m$ and G is isomorphic to a subgroup of the affine group $\mathrm{AGL}(m, p)$. But it is already known that a Hamiltonian 2-factorization of K_{p^m} with a group acting sharply transitively on vertices cannot exist if $m > 1$ (see [2], Proposition 4), then $m = 1$ and the assertion follows. □

This theorem gives a complete classification of all primitive Hamiltonian 2-factorization of K_v, while the non-Hamiltonian case remains an open problem. In the following paragraph we give an infinite family of examples and some non-existence results in this case.

3. The non-Hamiltonian case

Throughout this paragraph $\mathscr{F}$ will be a non-Hamiltonian 2-factorization of K_v with primitive automorphism group G . By Lemma 1 we have that v is not a prime in this case. When v is a genuine prime power an example for $\mathscr{F}$ is the 2-transitive 2-factorization given in [2]. It has to be noticed that in this case the full automorphism group is 2-transitive, however, it could contain a proper subgroup which is primitive but not 2-transitive on the vertex-set. Now, using a generalization of the technique of mixed translations given in [1], we construct examples of primitive 2-factorizations whose full automorphism group does not contain a 2-transitive subgroup.

3.1. Mixed translations over a finite field

Let F be a finite field of order p with $p \neq 2$ and, for $n \geqslant 3$, consider the n-dimensional vector space $V = F^n$. Identify the elements of V with the points of the affine space $\mathrm{AG}(n, p)$. Denote by t_a the translation of the affine space determined by the vector $\mathbf{a} \in V$ and by $T = \{t_a : \mathbf{a} \in V\}$ the translation group on $\mathrm{AG}(n, p)$.

Let W_0 be an hyperplane of $\mathrm{AG}(n, p)$ through $(0, \ldots, 0)$, and denote by $\overrightarrow{W_0}$ the vector subspace of V associated to W_0. Let $W_0, \ldots, W_{p-1}$ be the affine hyperplanes obtained by W_0 as $W_i = W_0 + i\mathbf{a}$, for $\mathbf{a} \notin \overrightarrow{W_0}$ and $i = 0, \ldots, p - 1$.

We will use mixed translations in what follows to obtain new examples of primitive 2-factorizations. For pairwise linearly independent vectors $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{p-1}$ in $\overrightarrow{W_0}$, these are transformations $m_{W_0, w_0, \ldots, w_{p-1}}$ defined by

$$m_{W_0,w_0,\ldots,w_{p-1}}(x) = \begin{cases} x + \mathbf{w}_0 & \text{if } x \in W_0, \\ x + \mathbf{w}_1 & \text{if } x \in W_1, \\ \vdots & \vdots \\ x + \mathbf{w}_{p-1} & \text{if } x \in W_{p-1}. \end{cases}$$

Note that $m_{W_0,w_0,\ldots,w_{p-1}}$ has a fixed-point-free action.

Identify the vertex-set of K_{p^n} with the point-set of the affine space $\mathrm{AG}(n, p)$. For each translation t_a of T we obtain a 2-factor whose cycles are obtained in the following way: take p^{n-1} points, $x_1, \ldots, x_{p^{n-1}}$, lying on different lines in the parallelism class generated by the vector $\mathbf{a}$. For each of these points, construct the cycle:

$$(x_j, x_j + \mathbf{a}, \ldots, x_j + (p - 1)\mathbf{a}),$$

where $j=1, \ldots, p^{n-1}$. A 2-factor is constructed in the same manner for each choice of the translation t_{ia}, $i=1, \ldots, p-1$. Obviously t_{ia} and t_{-ia} give rise to the same 2-factor and then we obtain $(p - 1)/2$ distinct 2-factors associated to the same class of parallel lines.

Analogously, $(p - 1)/2$ distinct 2-factors are generated from each mixed translation $m_{W_0,w_0,\ldots,w_{p-1}}$. For each hyperplane W_i ($i = 0, \ldots, p - 1$) we construct p^{n-2} p-cycles as follows: take p^{n-2} points, $x_1, \ldots, x_{p^{n-2}}$, of W_i which belong to distinct lines in the parallelism class generated by $\mathbf{w_i}$ on W_i, and construct the cycles:

$$(x_j, x_j + \mathbf{w_i}, \ldots, x_j + (p - 1)\mathbf{w_i}),$$

where $j = 1, \ldots, p^{n-2}$. As before $(p - 1)/2$ distinct 2-factors are constructed from $m_{W_0,kw_0,\ldots,kw_{p-1}}$, $k = 1, \ldots, p - 1$.

3.2. An example of a non-Hamiltonian primitive 2-factorization

Let d be a primitive divisor of p^n-1, that is a divisor of p^n-1 such that d does not divide p^m-1 for $m<n$. The existence of such a prime divisor is given by Zsigmondy's Lemma, see for instance [9].

Let B be the subgroup of order d of the multiplicative group $GF(p^n)^*$. Define G to be the group of all mappings $g: V \to V$ of the form $g(\mathbf{x}) = b \cdot \mathbf{x} + \mathbf{a}$ for some element $b \in B$ and some vector $\mathbf{a} \in V$. It is easy to prove that G acts primitively on V (see also [1]).

If we take p^n admitting a primitive divisor d, with $d \leqslant \lfloor (p^n-1)/p(p-1) \rfloor$, then a G-invariant 2-factorization of K_{p^n}, which is different from that arising from the standard line parallelism of $\mathrm{AG}(n,p)$ is constructed in the following proposition. There are infinitely many prime powers p^n which satisfy this requested condition on d: for example, if we take 3^n with $n \geqslant 3$ odd and such that $(3^n-1)/2$ is not a prime, then each prime d obtained by Zsigmondy's Lemma satisfies the hypothesis.

Proposition 1. *Let p^n, $n \geqslant 3$, be a prime power such that p^n-1 admits a primitive divisor d with $d \leqslant \lfloor (p^n-1)/p(p-1) \rfloor$. There exists a primitive 2-factorization of K_{p^n} which is not 2-transitive.*

Proof. Let G be the primitive group described above. This group has a primitive action on the points of $\mathrm{AG}(n,p)$. The action of G on the parallelism classes yields $(p^n-1)/d(p-1)$ orbits of the same length d. As $d \leqslant \lfloor (p^n-1)/p(p-1) \rfloor$ we have at least p distinct orbits. Let W_0 be an hyperplane of $\mathrm{AG}(n,p)$ through $(0,\ldots,0)$ and let $\mathbf{w_0},\ldots,\mathbf{w_{p-1}} \in \overrightarrow{W_0}$ be p vectors which determine p parallelism classes in distinct orbits under the action of G on $\mathrm{AG}(n,p)$. Consider the mixed translation $m_i = m_{W_0, iw_0,\ldots,iw_{p-1}}$, $i=1,\ldots,(p-1)/2$, and let F_i be the 2-factor arising from m_i. The group $T_{W_0} = \{t_a / \mathbf{a} \in \overrightarrow{W_0}\}$ is the stabilizer of each F_i in G and $|\mathrm{orb}_G(F_i)| = p^n d/(p^{n-1}) = pd$. The set $\bigcup_i \mathrm{orb}_G(F_i)$ contains $(p-1)/2pd$ 2-factors which are associated to pd distinct classes of parallelism. For each of the remaining $(p^n-1)/p-1-pd$ parallelism classes, take a vector $\mathbf{a}$ which determines the class, and construct the $(p-1)/2$ 2-factors arising from the translations t_{ia} $i=1,\ldots,(p-1)/2$ as described before. We obtain $(p^n-1)/2-(p-1)/2pd$ further 2-factors which, together with the $(p-1)/2pd$ 2-factors obtained from the mixed translations give rise to a 2-factorization $\mathscr{F}$ of K_{p^n} admitting G as an automorphism group acting primitively on the vertex-set. The full automorphism group of $\mathscr{F}$ does not contain a subgroup which is 2-transitive on the vertices: infact in this case each 2-factor should arise from a translation of T in the way described in Section 3.1, as proved in [2]. □

If the number of vertices v is not a prime power, no examples of primitive 2-factorizations of K_v are available.

3.3. Non-existence results

Now we give some necessary conditions for the existence of a primitive 2-factorization and then we will use these conditions to prove that K_9 has no primitive 2-factorizations different from the 2-transitive one and that K_{15} does not admit a primitive 2-factorization.

Lemma 2. *G does not fix a non-Hamiltonian factor of $\mathscr{F}$.*

Proof. The cycles of a non-Hamiltonian factor fixed by G form a system of imprimitivity for G. □

Lemma 3. *If v is not a prime, G does not fix a Hamiltonian factor of $\mathscr{F}$.*

Proof. Suppose G fixes a Hamiltonian factor F, then $G \leqslant D_v$. By the transitivity of G on the vertex-set and by the fact that v is odd, we also have $Z_v \leqslant G$. The transversals of a proper subgroup of Z_v form a system of imprimitivity for G. □

Lemma 4. *If G fixes a 2-factor, then v is a prime p, $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$ and either $G \cong \mathbb{Z}_p$ or $G \cong D_p$.*

Proof. This follows from Lemmas 1–3. □

Proposition 2. *A primitive* 2*-factorization of* K_9 *is isomorphic to the only* 2*-factorization* $\mathscr{F}$ *with* $\mathrm{Aut}(\mathscr{F}) \cong \mathrm{AGL}(2, 3)$, *and G is a subgroup of* $\mathrm{AGL}(2, 3)$ *acting primitively on the vertex-set.*

Proof. All 2-factorizations of K_9 and the order of the corresponding automorphism groups are enumerated in [8, Table 2, p. 437]. By the transitivity of G on the vertex-set, 9 must be a divisor of $|G|$ and only 4 possibilities remain. In the three cases where $\mathrm{Aut}(\mathscr{F})$ is different from $\mathrm{AGL}(2, 3)$ there is a factor fixed by $\mathrm{Aut}(\mathscr{F})$, therefore by Lemma 4 these cases are ruled out. The assertion follows. □

Proposition 3. *A primitive* 2*-factorization* $\mathscr{F}$ *of* K_{15} *does not exist.*

Proof. There are 6 groups admitting a primitive representation of degree 15: S_6, A_6, A_7, $\mathrm{PSL}(4, 2)$, S_{15} and A_{15}. The last three groups are at least 2-transitive on the vertex-set and then these cases are ruled out by the main theorem in [2]. The group A_6 is simple, its proper subgroups have index greater than 5, then the orbit-lengths under its action are equal to 1 or greater than 5. The first possibility is excluded by Lemma 4, then A_6 is transitive on the factors of $\mathscr{F}$, but the number of factors is not a divisor of the order of A_6: a contradiction. This implies that also S_6 is ruled out because it contains A_6. The only remaining possibility is A_7. Using the computer package GAP [10], we have checked that the pointwise stabilizer of 2 vertices is isomorphic to A_4 and fixes a further vertex. We can conclude that the 3–cycle, say T, with these 3 vertices belongs to $\mathscr{D}_{\mathscr{F}}$. Now, we have generated the only possible decomposition of K_{15} preserved by A_7 as the orbit of T under the action of the group A_7. The decomposition obtained is a Steiner triple system on 15 vertices and it is isomorphic to $PG(3, 2)$, see [11]. There are only two non-isomorphic parallelisms of $PG(3, 2)$, each having a group of order 168, and both intransitive (see for instance [5]). □

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